Equidistribution and Peyre's Constant

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1 Heights associated to adelic metrics

1.1 Motivation

The standard height function on $\mathbb{Q}$ is defined by

$$H(a/b) = \max\{|a|, |b|\},$$

where a/b is a fraction written in its lowest term.

Consider the natural embedding

$$\mathbb{Q} \to \mathbb{P}^1_{\mathbb{Q}}, \quad x \mapsto [x : 1].$$

We would like to define a height $H : \mathbb{P}^1_{\mathbb{Q}} \to \mathbb{R}$ extending the standard height on $\mathbb{Q}$. That is, we want

$$H([a/b : 1]) = \max\{|a|, |b|\}$$

for coprime integers a, b with $b \neq 0$. In other words, we want

$$H([a : b]) = \max\{|a|, |b|\}$$

for coprime integers a, b with $b \neq 0$. Therefore we define H by

$$H([a : b]) = \max\{|a|, |b|\}$$

for all coprime pairs of integers a, b, with b possibly 0. This generalises naturally to $\mathbb{P}^n_{\mathbb{Q}}$. Let $x \in \mathbb{P}^n_{\mathbb{Q}}$, and write $x = [x_0 : \ldots : x_n]$ for mutually coprime integers x_i. Then define

$$H(x) = \max_i\{|x_i|\}.$$

The condition on coprime integer coordinates is a bit annoying, so we would like a more intrinsic characterisation.

Lemma 1.1. *For mutually coprime integers $x_0, \ldots, x_n$, we have*

$$\max_i\{|x_i|\} = \prod_{v \in M_{\mathbb{Q}}} \max_i\{|x_i|_v\}.$$

Proof. Since the x_i are integers, we have

$$\max_i\{|x_i|_v\} = \begin{cases} \max_i|x_i| & \text{if } v = \infty, \\ 1 & \text{otherwise.} \end{cases}$$

□

Corollary 1.2. *Let $x \in \mathbb{P}^1_{\mathbb{Q}}$ and let $x = [x_0 : \ldots : x_n]$ be any choice of coordinates. Then we have*

$$H(x) = \prod_{v \in M_{\mathbb{Q}}} \max_i\{|x_i|_v\}.$$

Proof. Follows from the product formula, which tells us that

$$\prod_{v \in M_k} |\lambda|_v = 1$$

for all $\lambda \in \mathbb{Q}^\times$. □

This characterisation extends naturally to $\mathbb{P}^n_k$ for any number field k. Define $H_k : \mathbb{P}^n_k \to \mathbb{R}$ by

$$H_k(x) = \prod_{v \in M_k} \max_i\{|x_i|_v\}.$$

Let V be a variety over k and let $\mathcal{L}$ be a very ample sheaf. Let $s_0, \ldots, s_n \in \Gamma(X, \mathcal{L})$ be a basis, so that we have a closed embedding

$$\iota : V \hookrightarrow \mathbb{P}^n_k, \quad x \mapsto [s_0(x) : \ldots : s_n(x)].$$

Define $H_{\mathcal{L}} : V(k) \to \mathbb{R}$ by

$$H_{\mathcal{L}}(x) = H_k(\iota(x)) = \prod_{v \in M_k} \max_i \{|s_i(x)|_v\}.$$

Pick a section $s \in \Gamma(V, \mathcal{L})$, and define

$$\|s(x)\|_v := \min_i \left| \frac{s(x)}{s_i(x)} \right|_v,$$

where we adopt the convention that $\frac{1}{0} = \infty$ and $|\infty|_v = \infty$. Then we have

$$H_{\mathcal{L}}(x) = \prod_{v \in M_k} \|s(x)\|_v^{-1}.$$

The plan is to replace the collection $(\|\cdot\|_v)_{v \in M_k}$ by a more general notion called a **adelic metric**, and then define the height in terms of $\|\cdot\|_v$ in the same way.

1.2 Adelic metrics and their heights

Let $\mathcal{L}$ be a very ample sheaf on a Fano variety V over a number field k, and let $v \in M_k$. For $x \in V(k_v)$, we may pull $\mathcal{L}$ back to a sheaf on the base change V_{k_v}, which we also denote by $\mathcal{L}$. The sections $s \in \Gamma(V, \mathcal{L})$ take values $s(x)$ in a vector space

$$\mathcal{L}(x) = \mathcal{L}_x \otimes_{\mathcal{O}_{V_{k_v}, x}} k_v(x),$$

where $k_v(x)$ is the residue field of the local ring $\mathcal{O}_{V_{k_v}, x}$. A **v-adic metric** on $\mathcal{L}$ is a collection $\|\cdot\|_v := (\|\cdot\|_{v,x})_{x \in V(k_v)}$, such that for all $s \in \Gamma(V, \mathcal{L})$, the map

$$x \mapsto \|s(x)\|_{v,x}, \quad V(k_v) \to \mathbb{R}$$

is continuous. An **adelic metric** on V is a collection $(\|\cdot\|_v)_{v \in M_k}$ of v-adic metrics such that there exists a basis $s_0, \ldots, s_n$ of $\Gamma(V, \mathcal{L})$ such that for almost all $v \in M_f$, the following statement is true:

- For all $x \in V(k_v)$ and all $s \in \Gamma(V, \mathcal{L})$ with $s(x) \neq 0$, we have

$$\|s(x)\|_{v,x} = \min_i \left| \frac{s(x)}{s_i(x)} \right|_v .$$

Given an adelic metric $(\|\cdot\|_v)_{v \in M_k}$ on V, the height associated to it is

$$\mathbf{h}(x) = \prod_{x \in M_k} \|s(x)\|_{v,x}^{-1}.$$

Example 1.3. Defining

$$\|s(x)\|_{v,x} := \min_i \left| \frac{s(x)}{s_i(x)} \right|_v$$

for all places gives an adelic metric, and the associated height function is the usual height function corresponding to a basis of $\Gamma(V, \mathcal{L})$.

Example 1.4. Another adelic metric is given by

$$\|s(x)\|_{x,v} = \begin{cases} \min_i \left| \frac{s(x)}{s_i(x)} \right|_v & \text{if } v \in M_f, \\ \sqrt{\left| \frac{s_0(x)}{s(x)} \right|_v^2 + \ldots + \left| \frac{s_n(x)}{s(x)} \right|_v^2}^{-1} & \text{if } k_v \cong \mathbb{R}, \\ \left(\left| \frac{s_0(x)}{s(x)} \right|_v + \ldots + \left| \frac{s_n(x)}{s(x)} \right|_v \right)^{-1} & \text{if } k_v \cong \mathbb{C}. \end{cases}$$

Then this gives a genuinely different height function, which was used by Jeffrey Lin Thunder to count rational points on flag varieties.

2 Manin's Conjecture

2.1 Accumulating subvarieties and thin sets

Define

$$n_{\mathbf{h},V}(H) = \#\{P \in U(k) : \mathbf{h}(P) \leq H\}.$$

Ideally we would like to have

$$n_{\mathbf{h},V}(H) \sim CH(\log H)^{t-1}$$

for some constant C. Unfortunately this does not hold, and we have the following counterexample by Batyrev, Manin, Serre, et al.

Example 2.1. Let V be the projective plane $\mathbb{P}^2_{\mathbb{Q}}$ blown up at the point $P_0 = [0 : 0 : 1]$. Explicitly, we can write

$$V = \{([x : y : z], [s : t]) \in \mathbb{P}^2_{\mathbb{Q}} \times \mathbb{P}^1_{\mathbb{Q}} : xt = ys\}.$$

Let $\mathbf{h}$ be the height function

$$\mathbf{h}(P, Q) = \mathbf{h}_{\mathcal{O}_{\mathbb{P}^2_{\mathbb{Q}}}(1)}(P) \cdot \mathbf{h}_{\mathcal{O}_{\mathbb{P}^1_{\mathbb{Q}}}(1)}(Q)$$

Then we have

$$n_{\mathbf{h},V}(H) \sim \frac{2}{\zeta_{\mathbb{Q}}(2)} H^2,$$

so V does not satisfy the super naive conjecture. It turns out that the failure is "explained" by the exceptional divisor

$$E = \{([0 : 0 : 1], [s : t]) : [s : t] \in \mathbb{P}^1_{\mathbb{Q}}\}.$$

In particular, we have

$$n_{\mathbf{h},E}(H) \sim \frac{2}{\zeta_{\mathbb{Q}}(2)} H^2,$$

and

$$n_{\mathbf{h},V\setminus E}(H) \sim \frac{8}{3\zeta_{\mathbb{Q}}(2)^2} H \log H,$$

which is what we wanted.

This counterexample led Manin (I think) to define "accumulating subvarieties". Given a subset U of V, write

$$\beta_U = \limsup_{H\to\infty} \frac{\log n_U(H)}{\log H}.$$

This is essentially the power of H appearing in the asymptotic formula for $n_U(H)$.

Definition 2.2. An **accumulating subvariety** of V is a subvariety F of V such that for all nonempty open $W \subseteq F$, there is some nonempty open $U \subseteq V$ such that $\beta_W > \beta_U$.

The conjecture in Peyre is stated for the complements of all accumulating subvarieties. Unfortunately this also does not hold:

Example 2.3 (Batyrev-Tschinkel). Let V be the subvariety of $\mathbb{P}^3_{\mathbb{Q}} \times \mathbb{P}^3_{\mathbb{Q}}$ defined by the equation

$$\sum_{i=0}^{3} X_i Y_i^3 = 0.$$

There is a natural height function given by

$$\mathbf{h}(P, Q) = \mathbf{h}_{\mathcal{O}_{\mathbb{P}^3_{\mathbb{Q}}}(1)}(P) \cdot \mathbf{h}_{\mathcal{O}_{\mathbb{P}^3_{\mathbb{Q}}}(1)}(Q).$$

Let $\pi : V \to \mathbb{P}^3_{\mathbb{Q}}$ be the projection onto the first factor. For each $P = [x_0 : x_1 : x_2 : x_3] \in \mathbb{P}^3(\mathbb{Q})$, with all x_i nonzero, the fibre $V_P = \pi^{-1}(P)$ is a smooth cubic hypersurface in $\mathbb{P}^3_{\mathbb{Q}}$. Let U_P be the complement of the 27 lines in V_P. We can choose P such that $\operatorname{rank} \operatorname{Pic}(V_P) = 4$. Apparently, the conjecture is expected to be true for all dense open subsets of a smooth cubic surface, so for all dense open $U \subseteq U_P$ we have

$$n_{\mathbf{h},U}(H) \sim CH(\log H)^3.$$

On the other hand, apparently $\operatorname{rank} \operatorname{Pic}(V) = 2$, so the conjecture implies that

$$n_{\mathbf{h},U}(H) \sim CH(\log H),$$

for some dense open subset U. However, the set of P with $\operatorname{rank} \operatorname{Pic}(V_P) = 4$ is precisely the image of the map

$$c : \mathbb{P}^3 \to \mathbb{P}^3, \quad [x_0 : x_1 : x_2 : x_3] \mapsto [x_0^3 : x_1^3 : x_2^3 : x_3^3],$$

which is dense in $\mathbb{P}^3$. It follows that $U \cap U_P$ is nonempty for some such P, so

$$n_{\mathbf{h},U \cap U_P}(H) \sim CH(\log H)^3,$$

contradicting the conjectural asymptotics of $n_{\mathbf{h},U}(H)$.

This counterexample is explained by so-called **thin subsets**.

Definition 2.4. Let V be a nice variety. A subset T of $V(k)$ is called **thin** if it is in the image of a morphism $\varphi : X \to V$ of varieties that is generically finite and has no rational section.

The union of all badly behaved V_P above is thin. The crucial difference here is that thin sets are not necessarily Zariski closed, so the argument of the counterexample no longer applies.

Conjecture 2.5 (Manin's Conjecture)**.** Let V be a Fano variety over k with a very ample sheaf $\mathcal{L}$, and assume that $V(k)$ is dense in $V(\mathbb{A}_k)$. Then there exists some thin set $T \subseteq V(k)$ such that

$$n_{\mathbf{h},V\setminus T}(H) \sim CH(\log H)^{t-1}.$$

3 Peyre's constant

3.1 Local measures

Let V be a Fano variety over a number field k with very ample anticanonical sheaf. Fix an adelic measure on ω_V^{-1}. Let U be an open set of V with affine coordinates $x_1, \dots, x_n : U \to \mathbb{A}_k^n$. Somehow it seems like the differential operator

$$\frac{\partial}{\partial x_1} \wedge \dots \wedge \frac{\partial}{\partial x_n}$$

is in $\omega_V^{-1}(U)$, so its stalks are measured by the adelic metric. Using this as some kind of Jacobian, we get a measure

$$\boldsymbol{\omega}_v = \|\frac{\partial}{\partial x_1} \wedge \dots \wedge \frac{\partial}{\partial x_n}\|_v dx_{1,v} \dots dx_{n,v}.$$

on U, and we can glue to get a well-defined measure on V.

3.2 Local densities

For a finite set S of places of a number field k, write $\mathcal{O}_S$ for the set of **S-integers**

$$\mathcal{O}_S = \{x \in k : v(x) \geq 0 \text{ for all } v \notin S\}.$$

Lemma 3.1. *Let V be an n-dimensional Fano variety over a number field k, with very ample anticanonical bundle. There exists a finite set of places S of k and a geometrically smooth model $\mathscr{V}$ with geometrically integral fibres, such that for all $\mathfrak{p} \in M_k \setminus S$, we have*

1. *There is an isomorphism of Galois modules*

$$\operatorname{Pic}(V_{\bar{k}}) \to \operatorname{Pic}(\mathscr{V}_{\bar{\mathbb{F}}_\mathfrak{p}}).$$

2. *For all primes ℓ not divisible by $\mathfrak{p}$, the ℓ-primary part $\operatorname{Br}(\mathscr{V}_{\bar{\mathbb{F}}_\mathfrak{p}})[\ell^\infty]$ is finite.*

We will always write $\mathscr{V}$ and S for objects as in the lemma.

Definition 3.2. For $\mathfrak{p} \in M_{k,f} \setminus S$, the **local density** of V at $\mathfrak{p}$ is

$$d_\mathfrak{p}(V) = \frac{\#\mathscr{V}(\mathbb{F}_\mathfrak{p})}{N(\mathfrak{p})^{\dim V}}.$$

Lemma 3.3. *For almost all $\mathfrak{p} \in M_{k,f}$, we have*

$$d_\mathfrak{p}(V) = \boldsymbol{\omega}_\mathfrak{p}(V(k_\mathfrak{p})).$$

The product

$$\prod_{\mathfrak{p} \in M_{k,f} \setminus S} d_\mathfrak{p}(V)$$

is divergent, so the product

$$\prod_{\mathfrak{p} \in M_k} \omega_v(V(k_v))$$

is as well. The Frobenius $x \mapsto x^{N(\mathfrak{p})}$ on $\bar{\mathbb{F}}_\mathfrak{p}$ induces a morphism $\mathscr{V}_{\bar{\mathbb{F}}_\mathfrak{p}} \to \mathscr{V}_{\bar{\mathbb{F}}_\mathfrak{p}}$, which in turn induces an automorphism of $\operatorname{Pic} \mathscr{V}_{\bar{\mathbb{F}}_\mathfrak{p}}$. Write $\operatorname{Frob}_\mathfrak{p}$ for the inverse of this automorphism.

Call $\mathrm{Frob}_{\mathfrak{p}}$ the **geometric Frobenius morphism** on $\mathrm{Pic}\,\mathscr{V}_{\bar{\mathbb{F}}_{\mathfrak{p}}}$. Write $\overline{V} = V \times_k \bar{k}$. The local L-function associated to $\mathrm{Pic}\,\overline{V}$ at $\mathfrak{p}$ is

$$L_{\mathfrak{p}}(s, \mathrm{Pic}\,\overline{V}) = \frac{1}{\det(1 - N(\mathfrak{p})^{-s}\mathrm{Frob}_{\mathfrak{p}} \mid \mathrm{Pic}\,\mathscr{V}_{\bar{\mathbb{F}}_{\mathfrak{p}}} \otimes_{\mathbb{Z}} \mathbb{Q})}.$$

Lemma 3.4. *For all $\mathfrak{p} \in M_{k,f} \setminus S$, the quantity $L_{\mathfrak{p}}(1, \mathrm{Pic}\,\overline{V})$ is well-defined, and the product*

$$\prod_{\mathfrak{p} \in M_{k,f} \setminus S} \frac{d_{\mathfrak{p}}(V)}{L_{\mathfrak{p}}(1, \mathrm{Pic}\,\overline{V})}$$

converges absolutely.

3.3 Definition of the constant

Define real numbers λ_v by

$$\lambda_v = \begin{cases} L_v(1, \mathrm{Pic}\,\overline{V}) & \text{if } v \in M_{k,f} \setminus S, \\ 1 & \text{otherwise.} \end{cases}$$

Define the measure $\omega_{\mathbf{h},S}$ by

$$\omega_{\mathbf{h},S} = |\mathrm{disc}(k)|^{-\frac{\dim V}{2}} \cdot \prod_{v \in M_k} \frac{\omega_v}{\lambda_v}.$$

This measure depends on the model $\mathscr{V}$, and in particular on its set of primes of bad reduction S. We can make the measure independent of this information by defining

$$\omega_{\mathbf{h}} = \lim_{s \to 1} \big((s-1)^t L_S(s, \mathrm{Pic}\,\overline{V})\big)\omega_{\mathbf{h},S},$$

where L_S is given by

$$L_S(s, \mathrm{Pic}\,\overline{V}) = \prod_{\mathfrak{p} \in M_{k,f} \setminus S} L_{\mathfrak{p}}(s, \mathrm{Pic}\,\overline{V}).$$

Adding a prime to S multiplies the right-hand factor by $L_v(1, \mathrm{Pic}\,\overline{V})$ and divides the left-hand factor by the same thing, so $\omega_{\mathbf{h}}$ is well-defined.

Definition 3.5. The **Tamagawa constant** associated to $\mathbf{h}$ is

$$\tau_{\mathbf{h}} = \omega_{\mathbf{h}}(\overline{V(k)}).$$

Pick a basis $e_1, \ldots, e_t$ for $\operatorname{Pic} V$. There is a natural isomorphism of $\mathbb{R}$-vector spaces

$$\varphi : \operatorname{Pic} V \otimes \mathbb{R} \to \wedge^{t-1}(\operatorname{Pic} V \otimes \mathbb{R})^{\vee},$$

given by

$$e_i \mapsto e_1 \wedge \ldots \wedge e_{i-1} \wedge e_{i+1} \wedge \ldots e_t.$$

For $x \in \operatorname{Pic} V$, the element $\varphi(x)$ is in the subspace

$$\wedge^{t-1} x^{\perp} \subseteq \wedge^{t-1}(\operatorname{Pic} V \otimes \mathbb{R})^{\vee},$$

so $\varphi(x)$ is a $(t-1)$-form on the space $x^{\perp}$. Since $\dim x^{\perp}$ is $(t-1)$-dimensional, $\varphi(x)$ is a volume form on $x^{\perp}$. For all λ, the hyperplane

$$\mathscr{H}_x(\lambda) = \{y \in (\operatorname{Pic} V \otimes \mathbb{R})^{\vee} : y(x) = \lambda\}$$

is a translate of $x^{\perp}$, so $\varphi(x)$ defines a measure θ_x on $\mathscr{H}_x(\lambda)$. The above discussion only makes sense for $t \geq 2$. When $t = 1$, so that $\operatorname{Pic} V = \mathbb{Z} \cdot e_1$, we adopt the convention that

$$\theta_{me_1}(\mathscr{H}_{me_1}(\lambda)) = \frac{1}{m}$$

for all positive integers m and all $\lambda \in \mathbb{R}$.

Definition 3.6. The **dual effective cone** is

$$C_{\text{eff}}^{\vee}(V) = \{y \in (\operatorname{Pic} V \otimes \mathbb{R})^{\vee} : y([D]) \geq 0 \text{ for all effective } D \in \operatorname{Pic} V\}.$$

Definition 3.7. The **α-constant** is

$$\alpha(V) = \theta_{\omega_V^{-1}}(C_{\text{eff}}^{\vee}(V) \cap \mathscr{H}_{\omega_V^{-1}}(1)).$$

Definition 3.8. Peyre's constant $C_{\mathbf{h}}(V)$ is defined by

$$C_{\mathbf{h}}(V) = \alpha(V)\tau_{\mathbf{h}}(V).$$

3.4 Refined Manin conjecture

The following conjecture is as stated in [Pey95]. It does not mention thin sets, so it is known to be wrong.

Conjecture 3.9. Let V be a Fano variety with very ample anticanonical sheaf ω_V^{-1}, and let $\mathbf{h}$ be the height defined by an adelic metric on ω_V^{-1}. Assume that $V(k)$ is dense in $V(\mathbb{A}_k)$ and $\operatorname{Pic}\overline{V}$ is a "permutation module" of G_k. Then there is a dense open subset U of V such that

$$n_U(H) \sim C_{\mathbf{h}}(V)H(\log H)^{t-1},$$

where $t = \operatorname{rank}\operatorname{Pic} V$.

4 Independence from choice of metrics

Whenever we use the world "height", it will be relative to ω_V^{-1} and an adelic metric.

Fix an open subset $U \subseteq V$ and let $\mathbf{h}$ be a height. Define

$$n_{\mathbf{h},W}(H) := \#\{x \in U(k) \cap W : \mathbf{h}(x) \leq H\}.$$

Definition 4.1. A **good open set** is an open set $W \subseteq V(\mathbb{A}_k)$ such that there is a height $\mathbf{h}$ with

$$\omega_{\mathbf{h}}(\partial W) = 0.$$

Lemma 4.2. *If W is a good open set, then $\omega_{\mathbf{h}}(\partial W) = 0$ for all heights $\mathbf{h}$.*

Definition 4.3. We say that the rational points of V are **equidistributed on U** if there is a height function $\mathbf{h}$ such that for all good open sets W, we have

$$\lim_{H\to\infty} \frac{n_{\mathbf{h},W}(H)}{n_U(H)} = \frac{\omega_{\mathbf{h}}(\overline{V(k)} \cap W)}{\omega_{\mathbf{h}}(\overline{V(k)})}.$$

As shorthand, we say that V satisfies condition (E_U).

Theorem 4.4 ([Pey95, Proposition 3.3])**.**

1. *Suppose that V satisfies (E_U) for a height function $\mathbf{h}$. Then for all continuous $f : V(\mathbb{A}_k) \to \mathbb{R}$, we have*

$$\lim_{H\to\infty} \frac{\sum_{\{x\in U(k):\mathbf{h}(x)\leq H\}} f(x)}{n_U(H)} = \frac{\int_{\overline{V(k)}} f\omega_{\mathbf{h}}}{\omega_{\mathbf{h}}(\overline{V(k)})}.$$

2. *Let U be the complement of all accumulating subvarieties of V. Suppose that U is a Zariski open set defined over k and that V satisfies (E_U). Suppose that*

$$n_U(H) \sim C_{\mathbf{h}}(V)H(\log H)^{t-1}$$

for some height function $\mathbf{h}$. Then actually the formula holds for all choices of $\mathbf{h}$, i.e. the refined Manin conjecture for one height function implies the refined Manin conjecture for all height functions.

3. *If the refined Manin conjecture is true, then V satisfies (E_U), where U is the complement of all accumulating subvarieties.*

References

[Pey95] Emmanuel Peyre. "Hauteurs et mesures de Tamagawa sur les variétés de Fano". In: **Duke Mathematical Journal** 79.1 (1995), pp. 101–218. DOI: 10.1215/S0012-7094-95-07904-6. URL: https://doi.org/10.1215/S0012-7094-95-07904-6.